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Observable state space realizations for multivariable systems[☆]

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ABSTRACT

This paper derives two canonical state space forms (i.e., the observer canonical form and the observability canonical form) from multiple-input multiple-output systems described by difference equations. The state space model is expressed by the first-order difference equation and is equivalent to the input–output representation. More specifically, by setting the different state variables, the difference equations or the input–output representations can be transformed into two observable canonical forms and the canonical state space model can be also transformed into the difference equations. Finally, two examples are given.

1. Introduction

State space models play a significant role in system modeling and identification, adaptive control, and system analysis [1–5]. The state space model can describe a physical dynamic system with a set of the first-order differential equations or difference equations which involve the system input, output and state variables. The states represent the internal behavior of a system and the state space model maps the relationship from inputs to outputs. For multiple-input multiple-output systems (i.e., multivariable systems), the input and output are in a form of vectors. The state space representation provides a convenient and compact way to model and analyze systems. The process of transforming the differential equations (difference equations) or transfer function matrices to state space models is called the realization. The state-space realizations contain four basic canonical forms: (1) the observer canonical form and (2) the observability canonical form, which are observable (i.e., each state variable of the internal system can be completely determined by the output); (3) the controller canonical form and (4) the controllability canonical form, which are controllable (i.e., each state variable of the internal system can be completely determined by the input).

The difference equations are important because they act as mathematical models describing real life systems in computer science, economics, ecology [6–12]. In this literature, Kurbanli et al. studied the behavior of the positive solutions of the system of rational difference equations [13]; Iričanin discussed the global stability of some classes of higher-order nonlinear difference equations [14]. In the area of state space model identification, Schön et al. presented a system identification algorithm for the nonlinear state-space models [15]; Rachid et al. discussed the multivariable fractional system approximation with initial conditions using the integral state space representation [16]; Ding et al. presented the hierarchical state space model identification method for general dual-rate sampled-data systems [17] and for non-uniformly sampled-data systems [18–20].

Multivariable systems have received much attention in system analysis and control [21,22], system modeling and identification [23–32], and signal processing and parameter estimation [33–37]. Many methods discussed parameter

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identification problems of the input–output representations from state space models [38–48], including the multi-innovation identification algorithms [49–56], the maximum likelihood least squares algorithms [57–59], the iterative estimation algorithms [60–67] and the auxiliary model based identification algorithms [68–70]. This paper discusses the transformation relationships between the difference equations and the observer or observability canonical state space models for multivariable systems [17,18]. The proposed models could be applied to multivariable networked control system design and filtering [71–73].

Briefly, this paper is organized as follows. Sections 2 and 3 discuss the transformation between the difference equation and the observer canonical form. Sections 4 and 5 describe the transformation between the difference equation and the observability canonical form. Section 6 provides two examples to illustrate the proposed methods. Finally, we offer several concluding remarks in Section 7.

2. The observer canonical form from the difference equation

Systems with more than one input and more than one output are known as multiple-input multiple-output systems and the systems can be described by the difference equations. This section derives the state space model from a difference equation or an input–output representation.

Consider a linear dynamical system described by difference equation:

$$\begin{aligned}&\boldsymbol{y}(t)+\boldsymbol{A}_1\boldsymbol{y}(t-1)+\boldsymbol{A}_2\boldsymbol{y}(t-2)+\cdots+\boldsymbol{A}_n\boldsymbol{y}(t-n)\\&\quad=\boldsymbol{B}_1\boldsymbol{u}(t-1)+\boldsymbol{B}_2\boldsymbol{u}(t-2)+\boldsymbol{B}_3\boldsymbol{u}(t-3)+\cdots+\boldsymbol{B}_n\boldsymbol{u}(t-n),\end{aligned}\tag{1}$$

where $\boldsymbol{u}(t)\in\mathbb{R}^r$ and $\boldsymbol{y}(t)\in\mathbb{R}^m$ denote the input and output vectors of the system, t represents the time variable, $\boldsymbol{A}_i\in\mathbb{R}^{m\times m}$ and $\boldsymbol{B}_i\in\mathbb{R}^{m\times r}$ are coefficient matrices in z^{-1}.

Let $\boldsymbol{I}$ be an identity matrix of appropriate sizes and z be a unit forward shift operator: $z\boldsymbol{x}(t)=\boldsymbol{x}(t+1)$ or $z^{-1}\boldsymbol{x}(t)=\boldsymbol{x}(t-1)$, $\boldsymbol{A}(z)$ and $\boldsymbol{B}(z)$ are the matrix polynomials and are defined as

$$\begin{aligned}&\boldsymbol{A}(z):=\boldsymbol{I}+\boldsymbol{A}_1z^{-1}+\boldsymbol{A}_2z^{-2}+\cdots+\boldsymbol{A}_nz^{-n},\\&\boldsymbol{B}(z):=\boldsymbol{B}_1z^{-1}+\boldsymbol{B}_2z^{-2}+\boldsymbol{B}_3z^{-3}+\cdots+\boldsymbol{B}_nz^{-n}.\end{aligned}$$

Using the properties of the shift operator, Eq. (1) can be rewritten as

$$(\boldsymbol{I}+\boldsymbol{A}_1z^{-1}+\boldsymbol{A}_2z^{-2}+\cdots+\boldsymbol{A}_nz^{-n})\boldsymbol{y}(t)=(\boldsymbol{B}_1z^{-1}+\boldsymbol{B}_2z^{-2}+\boldsymbol{B}_3z^{-3}+\cdots+\boldsymbol{B}_nz^{-n})\boldsymbol{u}(t).$$

That is

$$\boldsymbol{A}(z)\boldsymbol{y}(t)=\boldsymbol{B}(z)\boldsymbol{u}(t),$$

or

$$\boldsymbol{y}(t)=\boldsymbol{A}^{-1}(z)\boldsymbol{B}(z)\boldsymbol{u}(t),\tag{2}$$

where transfer matrix $\boldsymbol{G}(z):=\boldsymbol{A}^{-1}(z)\boldsymbol{B}(z)\in\mathbb{R}^{m\times r}$ from the input $\boldsymbol{u}(t)$ to the output $\boldsymbol{y}(t)$ is a strictly proper rational fraction matrix.

The high-order difference equations are not convenience in applications and is often transformed into a set of the first-order difference equations. The details are as follows.

Define the state variables,

$$\boldsymbol{x}_1(t):=\boldsymbol{y}(t),$$

$$\begin{aligned}\boldsymbol{x}_2(t)&:=\boldsymbol{y}(t+1)+\boldsymbol{A}_1\boldsymbol{y}(t)-\boldsymbol{B}_1\boldsymbol{u}(t)\\&=\boldsymbol{x}_1(t+1)+\boldsymbol{A}_1\boldsymbol{x}_1(t)-\boldsymbol{B}_1\boldsymbol{u}(t),\end{aligned}\tag{3}$$

$$\begin{aligned}\boldsymbol{x}_3(t)&:=\boldsymbol{y}(t+2)+\boldsymbol{A}_1\boldsymbol{y}(t+1)+\boldsymbol{A}_2\boldsymbol{y}(t)-\boldsymbol{B}_1\boldsymbol{u}(t+1)-\boldsymbol{B}_2\boldsymbol{u}(t)\\&=\boldsymbol{x}_2(t+1)+\boldsymbol{A}_2\boldsymbol{x}_1(t)-\boldsymbol{B}_2\boldsymbol{u}(t),\end{aligned}\tag{4}$$

$$\vdots$$

$$\begin{aligned}\boldsymbol{x}_n(t)&:=\boldsymbol{y}(t+n-1)+\boldsymbol{A}_1\boldsymbol{y}(t+n-2)+\boldsymbol{A}_2\boldsymbol{y}(t+n-3)+\cdots+\boldsymbol{A}_{n-1}\boldsymbol{y}(t)\\&\qquad-\boldsymbol{B}_1\boldsymbol{u}(t+n-2)-\boldsymbol{B}_2\boldsymbol{u}(t+n-3)-\boldsymbol{B}_3\boldsymbol{u}(t+n-4)-\cdots-\boldsymbol{B}_{n-1}\boldsymbol{u}(t)\end{aligned}\tag{5}$$

$$=\boldsymbol{x}_{n-1}(t+1)+\boldsymbol{A}_{n-1}\boldsymbol{x}_1(t)-\boldsymbol{B}_{n-1}\boldsymbol{u}(t).\tag{6}$$

From Eqs. (3), (4) and (6), we have

$$\begin{cases}\boldsymbol{x}_1(t+1)=-\boldsymbol{A}_1\boldsymbol{x}_1(t)+\boldsymbol{x}_2(t)+\boldsymbol{B}_1\boldsymbol{u}(t),\\\boldsymbol{x}_2(t+1)=-\boldsymbol{A}_2\boldsymbol{x}_1(t)+\boldsymbol{x}_3(t)+\boldsymbol{B}_2\boldsymbol{u}(t),\\\vdots\\\boldsymbol{x}_{n-1}(t+1)=-\boldsymbol{A}_{n-1}\boldsymbol{x}_1(t)+\boldsymbol{x}_n(t)+\boldsymbol{B}_{n-1}\boldsymbol{u}(t).\end{cases}\tag{7}$$

Replacing t in (1) with $t+n$ gives

$$\begin{aligned}&\boldsymbol{y}(t+n)+\boldsymbol{A}_1\boldsymbol{y}(t+n-1)+\boldsymbol{A}_2\boldsymbol{y}(t+n-2)+\cdots+\boldsymbol{A}_{n-1}\boldsymbol{y}(t+1)\\&\quad-\boldsymbol{B}_1\boldsymbol{u}(t+n-1)-\boldsymbol{B}_2\boldsymbol{u}(t+n-2)-\boldsymbol{B}_3\boldsymbol{u}(t+n-3)-\cdots-\boldsymbol{B}_{n-1}\boldsymbol{u}(t+1)\\&=-\boldsymbol{A}_n\boldsymbol{y}(t)+\boldsymbol{B}_n\boldsymbol{u}(t).\end{aligned}$$

Replacing t in (6) with $t+1$ leads to the following equation,

$$\begin{aligned}\boldsymbol{x}_n(t+1)&=\boldsymbol{y}(t+n)+\boldsymbol{A}_1\boldsymbol{y}(t+n-1)+\boldsymbol{A}_2\boldsymbol{y}(t+n-2)+\cdots+\boldsymbol{A}_{n-1}\boldsymbol{y}(t+1)\\&\quad-\boldsymbol{B}_1\boldsymbol{u}(t+n-1)-\boldsymbol{B}_2\boldsymbol{u}(t+n-2)-\boldsymbol{B}_3\boldsymbol{u}(t+n-3)-\cdots-\boldsymbol{B}_{n-1}\boldsymbol{u}(t+1)\\&=-\boldsymbol{A}_n\boldsymbol{y}(t)+\boldsymbol{B}_n\boldsymbol{u}(t)\\&=-\boldsymbol{A}_n\boldsymbol{x}_1(t)+\boldsymbol{B}_n\boldsymbol{u}(t).\end{aligned}\tag{8}$$

Combining (7) and (8) gives the following state space model,

$$\begin{cases}\begin{bmatrix}\boldsymbol{x}_1(t+1)\\\boldsymbol{x}_2(t+1)\\\vdots\\\boldsymbol{x}_n(t+1)\end{bmatrix}=\begin{bmatrix}-\boldsymbol{A}_1&\boldsymbol{I}&\boldsymbol{0}&\cdots&\boldsymbol{0}\\-\boldsymbol{A}_2&\boldsymbol{0}&\boldsymbol{I}&\ddots&\vdots\\\vdots&\vdots&\ddots&\ddots&\boldsymbol{0}\\-\boldsymbol{A}_{n-1}&\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{I}\\-\boldsymbol{A}_n&\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1(t)\\\boldsymbol{x}_2(t)\\\vdots\\\boldsymbol{x}_n(t)\end{bmatrix}+\begin{bmatrix}\boldsymbol{B}_1\\\boldsymbol{B}_2\\\vdots\\\boldsymbol{B}_n\end{bmatrix}\boldsymbol{u}(t),\\[2ex]\boldsymbol{y}(t)=\boldsymbol{x}_1(t)=[\boldsymbol{I},\boldsymbol{0},\boldsymbol{0},\ldots,\boldsymbol{0}]\begin{bmatrix}\boldsymbol{x}_1(t)\\\boldsymbol{x}_2(t)\\\vdots\\\boldsymbol{x}_n(t)\end{bmatrix}.\end{cases}\tag{9}$$

Or

$$\begin{cases}\boldsymbol{x}_{\mathrm{o}}(t+1)=\boldsymbol{A}_{\mathrm{o}}\boldsymbol{x}_{\mathrm{o}}(t)+\boldsymbol{B}_{\mathrm{o}}\boldsymbol{u}(t),\\\boldsymbol{y}(t)=\boldsymbol{C}_{\mathrm{o}}\boldsymbol{x}_{\mathrm{o}}(t),\end{cases}\tag{10}$$

$$\boldsymbol{A}_{\mathrm{o}}:=\begin{bmatrix}-\boldsymbol{A}_1&\boldsymbol{I}&\boldsymbol{0}&\cdots&\boldsymbol{0}\\-\boldsymbol{A}_2&\boldsymbol{0}&\boldsymbol{I}&\ddots&\vdots\\\vdots&\vdots&\ddots&\ddots&\boldsymbol{0}\\-\boldsymbol{A}_{n-1}&\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{I}\\-\boldsymbol{A}_n&\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix}\in\mathbb{R}^{(mn)\times(mn)},\qquad\boldsymbol{B}_{\mathrm{o}}:=\begin{bmatrix}\boldsymbol{B}_1\\\boldsymbol{B}_2\\\vdots\\\boldsymbol{B}_n\end{bmatrix}\in\mathbb{R}^{(mn)\times r},$$

$$\boldsymbol{C}_{\mathrm{o}}:=[\boldsymbol{I},\boldsymbol{0},\boldsymbol{0},\ldots,\boldsymbol{0}]\in\mathbb{R}^{m\times(mn)},\qquad\boldsymbol{A}_i\in\mathbb{R}^{m\times m},\qquad\boldsymbol{B}_i\in\mathbb{R}^{m\times r},$$

where $\boldsymbol{x}_{\mathrm{o}}(t):=[\boldsymbol{x}_1(t),\boldsymbol{x}_2(t),\ldots,\boldsymbol{x}_n(t)]^{\mathrm{T}}\in\mathbb{R}^{mn}$ is the state vector, $\boldsymbol{A}_{\mathrm{o}}$ is the left-column block companion matrix [74,75], $\boldsymbol{B}_{\mathrm{o}}$ is an arbitrary matrix and $\boldsymbol{C}_{\mathrm{o}}$ is an matrix whose the first $m\times m$ block is an identity matrix and the rest is a zero matrix.

This state space model with $\boldsymbol{A}_{\mathrm{o}}$ and $\boldsymbol{C}_{\mathrm{o}}$ special structures is called the observer canonical form. The parameters in the observer canonical form are corresponding to that in the difference equation in (1) or the transfer function in (2).

3. The observer canonical form to the difference equation

This section is to transform the observer canonical form in (9) to its equivalent input–output representation. The basic idea is to eliminate the state vector $\boldsymbol{x}(t)$ in (9) and to obtain an input–output representation or difference equations.

Let

$$\boldsymbol{Q}:=z\boldsymbol{I}-\boldsymbol{A}_{\mathrm{o}}=\begin{bmatrix}z\boldsymbol{I}+\boldsymbol{A}_1&-\boldsymbol{I}&\boldsymbol{0}&\cdots&\boldsymbol{0}\\\boldsymbol{A}_2&z\boldsymbol{I}&-\boldsymbol{I}&\ddots&\vdots\\\vdots&\boldsymbol{0}&\ddots&\ddots&\boldsymbol{0}\\\boldsymbol{A}_{n-1}&\vdots&\ddots&z\boldsymbol{I}&-\boldsymbol{I}\\\boldsymbol{A}_n&\boldsymbol{0}&\cdots&\boldsymbol{0}&z\boldsymbol{I}\end{bmatrix},\qquad\boldsymbol{Q}^{-1}:=\begin{bmatrix}\boldsymbol{q}_{11}&\boldsymbol{q}_{12}&\boldsymbol{q}_{13}&\cdots&\boldsymbol{q}_{1n}\\*&*&*&\cdots&*\\\vdots&\vdots&\vdots&&\vdots\\*&*&*&\cdots&*\end{bmatrix}.$$

According to $\boldsymbol{Q}^{-1}\boldsymbol{Q} = \boldsymbol{I}$, we have

$$\begin{bmatrix} \boldsymbol{q}_{11} & \boldsymbol{q}_{12} & \boldsymbol{q}_{13} & \cdots & \boldsymbol{q}_{1n} \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \begin{bmatrix} z\boldsymbol{I}+\boldsymbol{A}_1 & -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{A}_2 & z\boldsymbol{I} & -\boldsymbol{I} & \ddots & \vdots \\ \vdots & \boldsymbol{0} & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}_{n-1} & \vdots & \ddots & z\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{A}_n & \boldsymbol{0} & \cdots & \boldsymbol{0} & z\boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} \end{bmatrix}.$$

Expanding this matrix equation, its first row gives the following equations:

$$\begin{cases} \boldsymbol{q}_{11}(z\boldsymbol{I}+\boldsymbol{A}_1) + \boldsymbol{q}_{12}\boldsymbol{A}_2 + \boldsymbol{q}_{13}\boldsymbol{A}_3 + \cdots + \boldsymbol{q}_{1n}\boldsymbol{A}_n = \boldsymbol{I}, \\ -\boldsymbol{q}_{11} + \boldsymbol{q}_{12}z = \boldsymbol{0}, \\ -\boldsymbol{q}_{12} + \boldsymbol{q}_{13}z = \boldsymbol{0}, \\ \quad\vdots \\ -\boldsymbol{q}_{1n-1} + \boldsymbol{q}_{1n}z = \boldsymbol{0}, \end{cases}$$

which can be equivalently rewritten as

$$\begin{cases} \boldsymbol{q}_{11} = (\boldsymbol{I}z^n + \boldsymbol{A}_1 z^{n-1} + \boldsymbol{A}_2 z^{n-2} + \cdots + \boldsymbol{A}_n)^{-1} z^{n-1} \\ \quad = (\boldsymbol{I} + \boldsymbol{A}_1 z^{-1} + \boldsymbol{A}_2 z^{-2} + \cdots + \boldsymbol{A}_n z^{-n})^{-1} z^{-1} \\ \quad = \boldsymbol{A}^{-1}(z) z^{-1}, \\ \boldsymbol{q}_{12} = \boldsymbol{q}_{11} z^{-1}, \\ \boldsymbol{q}_{13} = \boldsymbol{q}_{11} z^{-2}, \\ \vdots \\ \boldsymbol{q}_{1n} = \boldsymbol{q}_{11} z^{-n+1}. \end{cases}$$

Using the properties of the shift operator, the first equation of (10) gives

$$z\boldsymbol{x}_{\mathrm{o}}(t) = \boldsymbol{A}_{\mathrm{o}}\boldsymbol{x}_{\mathrm{o}}(t) + \boldsymbol{B}_{\mathrm{o}}\boldsymbol{u}(t),$$

or

$$\boldsymbol{x}_{\mathrm{o}}(t) = (z\boldsymbol{I} - \boldsymbol{A}_{\mathrm{o}})^{-1}\boldsymbol{B}_{\mathrm{o}}\boldsymbol{u}(t).$$

Substituting $\boldsymbol{x}_{\mathrm{o}}(t)$ into the second equation of (10) gives

$$\boldsymbol{y}(t) = \boldsymbol{C}_{\mathrm{o}}(z\boldsymbol{I} - \boldsymbol{A}_{\mathrm{o}})^{-1}\boldsymbol{B}_{\mathrm{o}}\boldsymbol{u}(t).$$

Thus we have the transfer matrix from the input $\boldsymbol{u}(t)$ to the output $\boldsymbol{y}(t)$:

$$\begin{aligned} \boldsymbol{G}(z) &= \boldsymbol{C}_{\mathrm{o}}(z\boldsymbol{I} - \boldsymbol{A}_{\mathrm{o}})^{-1}\boldsymbol{B}_{\mathrm{o}} \\ &= [\boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0}] \begin{bmatrix} z\boldsymbol{I}+\boldsymbol{A}_1 & -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{A}_2 & z\boldsymbol{I} & -\boldsymbol{I} & \ddots & \vdots \\ \vdots & \boldsymbol{0} & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{A}_{n-1} & \vdots & \ddots & z\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{A}_n & \boldsymbol{0} & \cdots & \boldsymbol{0} & z\boldsymbol{I} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_n \end{bmatrix} \\ &= [\boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0}] \begin{bmatrix} \boldsymbol{q}_{11} & \boldsymbol{q}_{12} & \boldsymbol{q}_{13} & \cdots & \boldsymbol{q}_{1n} \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_n \end{bmatrix} \\ &= \boldsymbol{q}_{11}\boldsymbol{B}_1 + \boldsymbol{q}_{12}\boldsymbol{B}_2 + \boldsymbol{q}_{13}\boldsymbol{B}_3 + \cdots + \boldsymbol{q}_{1n}\boldsymbol{B}_n \\ &= \boldsymbol{q}_{11}\boldsymbol{B}_1 + \boldsymbol{q}_{11}\boldsymbol{B}_2 z^{-1} + \boldsymbol{q}_{11}\boldsymbol{B}_3 z^{-2} + \cdots + \boldsymbol{q}_{11}\boldsymbol{B}_n z^{-n+1} \\ &= \boldsymbol{q}_{11}(\boldsymbol{B}_1 + \boldsymbol{B}_2 z^{-1} + \boldsymbol{B}_3 z^{-2} + \cdots + \boldsymbol{B}_n z^{-n+1}) \\ &= \boldsymbol{A}^{-1}(z) z^{-1}(\boldsymbol{B}_1 + \boldsymbol{B}_2 z^{-1} + \boldsymbol{B}_3 z^{-2} + \cdots + \boldsymbol{B}_n z^{-n+1}) \\ &= \boldsymbol{A}^{-1}(z)\boldsymbol{B}(z). \end{aligned}$$

Hence, we have the input–output representation:

$$\boldsymbol{A}(z)\boldsymbol{y}(t) = \boldsymbol{B}(z)\boldsymbol{u}(t).$$

This is equivalent to the difference equation in (1).

4. The observability canonical form from the difference equation

This section derives the observability canonical form from (1). Define new state variables,

$$\boldsymbol{x}_1(t) := \boldsymbol{y}(t), \tag{11}$$

$$\boldsymbol{x}_2(t) := \boldsymbol{y}(t+1) - \boldsymbol{\beta}_1\boldsymbol{u}(t), \tag{12}$$

$$\boldsymbol{x}_3(t) := \boldsymbol{y}(t+2) - \boldsymbol{\beta}_1\boldsymbol{u}(t+1) - \boldsymbol{\beta}_2\boldsymbol{u}(t), \tag{13}$$

$$\vdots$$

$$\boldsymbol{x}_n(t) := \boldsymbol{y}(t+n-1) - \boldsymbol{\beta}_1\boldsymbol{u}(t+n-2) - \boldsymbol{\beta}_2\boldsymbol{u}(t+n-3) - \cdots - \boldsymbol{\beta}_{n-1}\boldsymbol{u}(t), \tag{14}$$

$$\boldsymbol{x}_n(t+1) := \boldsymbol{y}(t+n) - \boldsymbol{\beta}_1\boldsymbol{u}(t+n-1) - \boldsymbol{\beta}_2\boldsymbol{u}(t+n-2) - \cdots - \boldsymbol{\beta}_{n-1}\boldsymbol{u}(t+1). \tag{15}$$

Subtracting $\boldsymbol{\beta}_n\boldsymbol{u}(t)$ from both sides of (15) gives

$$\boldsymbol{x}_n(t+1) - \boldsymbol{\beta}_n\boldsymbol{u}(t) = \boldsymbol{y}(t+n) - \boldsymbol{\beta}_1\boldsymbol{u}(t+n-1) - \boldsymbol{\beta}_2\boldsymbol{u}(t+n-2) - \cdots - \boldsymbol{\beta}_{n-1}\boldsymbol{u}(t+1) - \boldsymbol{\beta}_n\boldsymbol{u}(t). \tag{16}$$

Pre-multiplying (11)–(14) by the matrices $\boldsymbol{A}_n, \boldsymbol{A}_{n-1}, \boldsymbol{A}_{n-2}$ and $\boldsymbol{A}_1$, respectively, gives

$$\begin{cases} \boldsymbol{A}_n\boldsymbol{x}_1(t) = \boldsymbol{A}_n\boldsymbol{y}(t), \\ \boldsymbol{A}_{n-1}\boldsymbol{x}_2(t) = \boldsymbol{A}_{n-1}\boldsymbol{y}(t+1) - \boldsymbol{A}_{n-1}\boldsymbol{\beta}_1\boldsymbol{u}(t), \\ \boldsymbol{A}_{n-2}\boldsymbol{x}_3(t) = \boldsymbol{A}_{n-2}\boldsymbol{y}(t+2) - \boldsymbol{A}_{n-2}\boldsymbol{\beta}_1\boldsymbol{u}(t+1) - \boldsymbol{A}_{n-2}\boldsymbol{\beta}_2\boldsymbol{u}(t), \\ \vdots \\ \boldsymbol{A}_1\boldsymbol{x}_n(t) = \boldsymbol{A}_1\boldsymbol{y}(t+n-1) - \boldsymbol{A}_1\boldsymbol{\beta}_1\boldsymbol{u}(t+n-2) - \boldsymbol{A}_1\boldsymbol{\beta}_2\boldsymbol{u}(t+n-3) - \cdots - \boldsymbol{A}_1\boldsymbol{\beta}_{n-1}\boldsymbol{u}(t). \end{cases} \tag{17}$$

Combining (16) and (17) gives

$$\begin{cases} \boldsymbol{A}_n\boldsymbol{y}(t) = \boldsymbol{A}_n\boldsymbol{x}_1(t), \\ \boldsymbol{A}_{n-1}\boldsymbol{y}(t+1) = \boldsymbol{A}_{n-1}\boldsymbol{x}_2(t) + \boldsymbol{A}_{n-1}\boldsymbol{\beta}_1\boldsymbol{u}(t), \\ \boldsymbol{A}_{n-2}\boldsymbol{y}(t+2) = \boldsymbol{A}_{n-2}\boldsymbol{x}_3(t) + \boldsymbol{A}_{n-2}\boldsymbol{\beta}_1\boldsymbol{u}(t+1) + \boldsymbol{A}_{n-2}\boldsymbol{\beta}_2\boldsymbol{u}(t), \\ \vdots \\ \boldsymbol{A}_1\boldsymbol{y}(t+n-1) = \boldsymbol{A}_1\boldsymbol{x}_n(t) + \boldsymbol{A}_1\boldsymbol{\beta}_1\boldsymbol{u}(t+n-2) + \boldsymbol{A}_1\boldsymbol{\beta}_2\boldsymbol{u}(t+n-3) + \cdots + \boldsymbol{A}_1\boldsymbol{\beta}_{n-1}\boldsymbol{u}(t), \\ \boldsymbol{y}(t+n) = \boldsymbol{x}_n(t+1) - \boldsymbol{\beta}_n\boldsymbol{u}(t) + \boldsymbol{\beta}_1\boldsymbol{u}(t+n-1) + \boldsymbol{\beta}_2\boldsymbol{u}(t+n-2) + \cdots + \boldsymbol{\beta}_n\boldsymbol{u}(t). \end{cases}$$

Adding the above equations gives

$$\begin{aligned} &\boldsymbol{y}(t+n) + \boldsymbol{A}_1\boldsymbol{y}(t+n-1) + \boldsymbol{A}_2\boldsymbol{y}(t+n-2) + \cdots + \boldsymbol{A}_n\boldsymbol{y}(t) \\ &\quad = \boldsymbol{x}_n(t+1) - \boldsymbol{\beta}_n\boldsymbol{u}(t) + \boldsymbol{A}_1\boldsymbol{x}_n(t) + \boldsymbol{A}_2\boldsymbol{x}_{n-1}(t) + \cdots + \boldsymbol{A}_n\boldsymbol{x}_1(t) \\ &\qquad + \boldsymbol{\beta}_1\boldsymbol{u}(t+n-1) + (\boldsymbol{A}_1\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)\boldsymbol{u}(t+n-2) + (\boldsymbol{A}_2\boldsymbol{\beta}_1 + \boldsymbol{A}_1\boldsymbol{\beta}_2 + \boldsymbol{\beta}_3)\boldsymbol{u}(t+n-3) \\ &\qquad + \cdots + (\boldsymbol{A}_{n-1}\boldsymbol{\beta}_1 + \boldsymbol{A}_{n-2}\boldsymbol{\beta}_2 + \boldsymbol{A}_{n-3}\boldsymbol{\beta}_3 + \cdots + \boldsymbol{\beta}_n)\boldsymbol{u}(t). \end{aligned}$$

Let

$$\begin{cases} \boldsymbol{B}_1 := \boldsymbol{\beta}_1, \\ \boldsymbol{B}_2 := \boldsymbol{A}_1\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2, \\ \boldsymbol{B}_3 := \boldsymbol{A}_2\boldsymbol{\beta}_1 + \boldsymbol{A}_1\boldsymbol{\beta}_2 + \boldsymbol{\beta}_3, \\ \vdots \\ \boldsymbol{B}_n := \boldsymbol{A}_{n-1}\boldsymbol{\beta}_1 + \boldsymbol{A}_{n-2}\boldsymbol{\beta}_2 + \boldsymbol{A}_{n-3}\boldsymbol{\beta}_3 + \cdots + \boldsymbol{\beta}_n, \end{cases}$$

which is equivalent to

$$\begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & & & & \\ \boldsymbol{A}_1 & \boldsymbol{I} & & & \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{I} & & \\ \vdots & & \ddots & \ddots & \\ \boldsymbol{A}_{n-1} & \boldsymbol{A}_{n-2} & \cdots & \boldsymbol{A}_1 & \boldsymbol{I} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_n \end{bmatrix}.$$

Replacing t in (1) with $t+n$ yields

$$\begin{aligned} &\boldsymbol{y}(t+n) + \boldsymbol{A}_1\boldsymbol{y}(t+n-1) + \boldsymbol{A}_2\boldsymbol{y}(t+n-2) + \cdots + \boldsymbol{A}_n\boldsymbol{y}(t) \\ &\quad = \boldsymbol{B}_1\boldsymbol{u}(t+n-1) + \boldsymbol{B}_2\boldsymbol{u}(t+n-2) + \boldsymbol{B}_3\boldsymbol{u}(t+n-3) + \cdots + \boldsymbol{B}_n\boldsymbol{u}(t). \end{aligned}$$

Adding Eq. (16) to (17) gives

$$\begin{aligned}
&\boldsymbol{x}_n(t+1)-\boldsymbol{\beta}_n\boldsymbol{u}(t)+\boldsymbol{A}_1\boldsymbol{x}_n(t)+\boldsymbol{A}_2\boldsymbol{x}_{n-1}(t)+\cdots+\boldsymbol{A}_n\boldsymbol{x}_1(t)\\
&\quad=\boldsymbol{y}(t+n)-\boldsymbol{\beta}_1\boldsymbol{u}(t+n-1)-\boldsymbol{\beta}_2\boldsymbol{u}(t+n-2)-\cdots-\boldsymbol{\beta}_n\boldsymbol{u}(t)\\
&\qquad+\boldsymbol{A}_1[\boldsymbol{y}(t+n-1)-\boldsymbol{\beta}_1\boldsymbol{u}(t+n-2)-\boldsymbol{\beta}_2\boldsymbol{u}(t+n-3)-\cdots-\boldsymbol{\beta}_{n-1}\boldsymbol{u}(t)]\\
&\qquad+\boldsymbol{A}_2[\boldsymbol{y}(t+n-2)-\boldsymbol{\beta}_1\boldsymbol{u}(t+n-3)-\boldsymbol{\beta}_2\boldsymbol{u}(t+n-4)-\cdots-\boldsymbol{\beta}_{n-2}\boldsymbol{u}(t)]+\cdots+\boldsymbol{A}_n\boldsymbol{y}(t)\\
&\quad=\boldsymbol{y}(t+n)+\boldsymbol{A}_1\boldsymbol{y}(t+n-1)+\boldsymbol{A}_2\boldsymbol{y}(t+n-2)+\cdots+\boldsymbol{A}_n\boldsymbol{y}(t)\\
&\qquad-\boldsymbol{\beta}_1\boldsymbol{u}(t+n-1)-(\boldsymbol{A}_1\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)\boldsymbol{u}(t+n-2)-(\boldsymbol{A}_2\boldsymbol{\beta}_1+\boldsymbol{A}_1\boldsymbol{\beta}_2+\boldsymbol{\beta}_3)\boldsymbol{u}(t+n-3)\\
&\qquad-\cdots-(\boldsymbol{A}_{n-1}\boldsymbol{\beta}_1+\boldsymbol{A}_{n-2}\boldsymbol{\beta}_2+\boldsymbol{A}_{n-3}\boldsymbol{\beta}_3+\cdots+\boldsymbol{\beta}_n)\boldsymbol{u}(t)\\
&\quad=\boldsymbol{y}(t+n)+\boldsymbol{A}_1\boldsymbol{y}(t+n-1)+\boldsymbol{A}_2\boldsymbol{y}(t+n-2)+\cdots+\boldsymbol{A}_n\boldsymbol{y}(t)\\
&\qquad-\boldsymbol{B}_1\boldsymbol{u}(t+n-1)-\boldsymbol{B}_2\boldsymbol{u}(t+n-2)-\boldsymbol{B}_3\boldsymbol{u}(t+n-3)-\cdots-\boldsymbol{B}_n\boldsymbol{u}(t)\\
&\quad=\boldsymbol{0},
\end{aligned}$$

or

$$\boldsymbol{x}_n(t+1)-\boldsymbol{\beta}_n\boldsymbol{u}(t)=-\boldsymbol{A}_n\boldsymbol{x}_1(t)-\boldsymbol{A}_{n-1}\boldsymbol{x}_2(t)-\boldsymbol{A}_{n-2}\boldsymbol{x}_3(t)-\cdots-\boldsymbol{A}_1\boldsymbol{x}_n(t). \tag{18}$$

Replacing t in (11)–(14) with $t+1$ and combining (18) gives

$$\begin{cases}
\boldsymbol{x}_1(t+1)=\boldsymbol{y}(t+1)\\
\quad=\boldsymbol{x}_2(t)+\boldsymbol{\beta}_1\boldsymbol{u}(t),\\
\boldsymbol{x}_2(t+1)=\boldsymbol{y}(t+2)-\boldsymbol{\beta}_1\boldsymbol{u}(t+1)\\
\quad=\boldsymbol{x}_3(t)+\boldsymbol{\beta}_2\boldsymbol{u}(t),\\
\vdots\\
\boldsymbol{x}_{n-1}(t+1)=\boldsymbol{y}(t+n-1)-\boldsymbol{\beta}_1\boldsymbol{u}(t+n-2)-\boldsymbol{\beta}_2\boldsymbol{u}(t+n-3)-\cdots-\boldsymbol{\beta}_{n-2}\boldsymbol{u}(t+1)\\
\quad=\boldsymbol{x}_n(t)+\boldsymbol{\beta}_{n-1}\boldsymbol{u}(t),\\
\boldsymbol{x}_n(t+1)=-\boldsymbol{A}_n\boldsymbol{x}_1(t)-\boldsymbol{A}_{n-1}\boldsymbol{x}_2(t)-\boldsymbol{A}_{n-2}\boldsymbol{x}_3(t)-\cdots-\boldsymbol{A}_1\boldsymbol{x}_n(t)+\boldsymbol{\beta}_n\boldsymbol{u}(t).
\end{cases}$$

Combining (11) and the above equations gives the following state space model,

$$\begin{cases}
\begin{bmatrix}\boldsymbol{x}_1(t+1)\\ \boldsymbol{x}_2(t+1)\\ \vdots\\ \boldsymbol{x}_n(t+1)\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\cdots&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\ddots&\vdots\\ \vdots&\vdots&\ddots&\ddots&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{I}\\ -\boldsymbol{A}_n&-\boldsymbol{A}_{n-1}&-\boldsymbol{A}_{n-2}&\cdots&-\boldsymbol{A}_1\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_1(t)\\ \boldsymbol{x}_2(t)\\ \vdots\\ \boldsymbol{x}_n(t)\end{bmatrix}+\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \vdots\\ \boldsymbol{\beta}_n\end{bmatrix}\boldsymbol{u}(t),\\
\boldsymbol{y}(t)=[\boldsymbol{I},\boldsymbol{0},\boldsymbol{0},\ldots,\boldsymbol{0}]\begin{bmatrix}\boldsymbol{x}_1(t)\\ \boldsymbol{x}_2(t)\\ \vdots\\ \boldsymbol{x}_n(t)\end{bmatrix},
\end{cases} \tag{19}$$

where

$$\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \vdots\\ \boldsymbol{\beta}_n\end{bmatrix}=\begin{bmatrix}\boldsymbol{I}&&&&\\ \boldsymbol{A}_1&\boldsymbol{I}&&&\\ \boldsymbol{A}_2&\boldsymbol{A}_1&\boldsymbol{I}&&\\ \vdots&&\ddots&\ddots&\\ \boldsymbol{A}_{n-1}&\boldsymbol{A}_{n-2}&\cdots&\boldsymbol{A}_1&\boldsymbol{I}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{B}_1\\ \boldsymbol{B}_2\\ \vdots\\ \boldsymbol{B}_n\end{bmatrix}.$$

Or

$$\begin{cases}\boldsymbol{x}_{\text{ob}}(t+1)=\boldsymbol{A}_{\text{ob}}\boldsymbol{x}_{\text{ob}}(t)+\boldsymbol{B}_{\text{ob}}\boldsymbol{u}(t),\\ \boldsymbol{y}(t)=\boldsymbol{C}_{\text{ob}}\boldsymbol{x}_{\text{ob}}(t),\end{cases}$$

$$\boldsymbol{A}_{\text{ob}}:=\begin{bmatrix}\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\cdots&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\ddots&\vdots\\ \vdots&\vdots&\ddots&\ddots&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{I}\\ -\boldsymbol{A}_n&-\boldsymbol{A}_{n-1}&-\boldsymbol{A}_{n-2}&\cdots&-\boldsymbol{A}_1\end{bmatrix}\in\mathbb{R}^{(mn)\times(mn)},\qquad \boldsymbol{B}_{\text{ob}}:=\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \vdots\\ \boldsymbol{\beta}_n\end{bmatrix}\in\mathbb{R}^{(mn)\times r},$$

$$\boldsymbol{C}_{\text{ob}}:=[\boldsymbol{I},\boldsymbol{0},\boldsymbol{0},\ldots,\boldsymbol{0}]\in\mathbb{R}^{m\times(mn)},\qquad \boldsymbol{A}_i\in\mathbb{R}^{m\times m},\qquad \boldsymbol{\beta}_i\in\mathbb{R}^{m\times r},$$

where $\boldsymbol{x}_{\mathrm{ob}}(t) := [\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \ldots, \boldsymbol{x}_n(t)]^{\mathrm{T}} \in \mathbb{R}^{mn}$ is the state vector, $\boldsymbol{A}_{\mathrm{ob}}$ is the bottom-row companion matrix, $\boldsymbol{B}_{\mathrm{ob}}$ is an arbitrary matrix and $\boldsymbol{C}_{\mathrm{ob}}$ is an matrix whose the first $m \times m$ block is an identity matrix and the rest is a zero matrix.

This state space model with $\boldsymbol{A}_{\mathrm{ob}}$ and $\boldsymbol{C}_{\mathrm{ob}}$ special structures is called the observability canonical form.

5. The observability canonical form to the difference equation

In this section, from the observability canonical form in (19), we derive its input–output representation, which does not involve the state vector $\boldsymbol{x}(t)$.

Let

$$\boldsymbol{P} := z\boldsymbol{I} - \boldsymbol{A}_{\mathrm{ob}} = \begin{bmatrix} z\boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & z\boldsymbol{I} & -\boldsymbol{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & z\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & \cdots & \boldsymbol{A}_2 & z\boldsymbol{I} + \boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{P}^{-1} := \begin{bmatrix} \boldsymbol{p}_{11} & \boldsymbol{p}_{12} & \boldsymbol{p}_{13} & \cdots & \boldsymbol{p}_{1n} \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix}.$$

According to $\boldsymbol{P}^{-1}\boldsymbol{P} = \boldsymbol{I}$, we have

$$\begin{bmatrix} \boldsymbol{p}_{11} & \boldsymbol{p}_{12} & \boldsymbol{p}_{13} & \cdots & \boldsymbol{p}_{1n} \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \begin{bmatrix} z\boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & z\boldsymbol{I} & -\boldsymbol{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & z\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & \cdots & \boldsymbol{A}_2 & z\boldsymbol{I} + \boldsymbol{A}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} \end{bmatrix}.$$

Expanding this matrix equation, its first row gives the following equations:

$$\begin{cases} \boldsymbol{p}_{11}z + \boldsymbol{p}_{1n}\boldsymbol{A}_n = \boldsymbol{I}, \\ -\boldsymbol{p}_{11} + \boldsymbol{p}_{12}z + \boldsymbol{p}_{1n}\boldsymbol{A}_{n-1} = \boldsymbol{0}, \\ -\boldsymbol{p}_{12} + \boldsymbol{p}_{13}z + \boldsymbol{p}_{1n}\boldsymbol{A}_{n-2} = \boldsymbol{0}, \\ \quad\vdots \\ -\boldsymbol{p}_{1n-1} + \boldsymbol{p}_{1n}(z\boldsymbol{I} + \boldsymbol{A}_1) = \boldsymbol{0}, \end{cases}$$

which can be equivalently rewritten as

$$\begin{aligned} &\boldsymbol{p}_{1n} = (\boldsymbol{I}z^n + \boldsymbol{A}_1 z^{n-1} + \boldsymbol{A}_2 z^{n-2} + \cdots + \boldsymbol{A}_n)^{-1}, \\ &\boldsymbol{p}_{1n-1} = \boldsymbol{p}_{1n}(\boldsymbol{I}z + \boldsymbol{A}_1), \\ &\vdots \\ &\boldsymbol{p}_{13} = \boldsymbol{p}_{1n}(\boldsymbol{I}z^{n-3} + \boldsymbol{A}_1 z^{n-4} + \boldsymbol{A}_2 z^{n-5} + \cdots + \boldsymbol{A}_{n-3}), \\ &\boldsymbol{p}_{12} = \boldsymbol{p}_{1n}(\boldsymbol{I}z^{n-2} + \boldsymbol{A}_1 z^{n-3} + \boldsymbol{A}_2 z^{n-4} + \cdots + \boldsymbol{A}_{n-2}), \\ &\boldsymbol{p}_{11} = \boldsymbol{p}_{1n}(\boldsymbol{I}z^{n-1} + \boldsymbol{A}_1 z^{n-2} + \boldsymbol{A}_2 z^{n-3} + \cdots + \boldsymbol{A}_{n-1}). \end{aligned}$$

The transfer matrix from the input $\boldsymbol{u}(t)$ to the output $\boldsymbol{y}(t)$ is

$$\begin{aligned} \boldsymbol{G}(z) &= \boldsymbol{C}_{\mathrm{ob}}(z\boldsymbol{I} - \boldsymbol{A}_{\mathrm{ob}})^{-1}\boldsymbol{B}_{\mathrm{ob}} \\ &= [\boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0}] \begin{bmatrix} z\boldsymbol{I} & -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & z\boldsymbol{I} & -\boldsymbol{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & z\boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & \cdots & \boldsymbol{A}_2 & z\boldsymbol{I} + \boldsymbol{A}_1 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_n \end{bmatrix} \\ &= [\boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}, \ldots, \boldsymbol{0}] \begin{bmatrix} \boldsymbol{p}_{11} & \boldsymbol{p}_{12} & \boldsymbol{p}_{13} & \cdots & \boldsymbol{p}_{1n} \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_n \end{bmatrix} \\ &= \boldsymbol{p}_{11}\boldsymbol{\beta}_1 + \boldsymbol{p}_{12}\boldsymbol{\beta}_2 + \boldsymbol{p}_{13}\boldsymbol{\beta}_3 + \cdots + \boldsymbol{p}_{1n}\boldsymbol{\beta}_n \\ &= \boldsymbol{p}_{1n}[(\boldsymbol{I}z^{n-1} + \boldsymbol{A}_1 z^{n-2} + \boldsymbol{A}_2 z^{n-3} + \cdots + \boldsymbol{A}_{n-1})\boldsymbol{\beta}_1 + (\boldsymbol{I}z^{n-2} + \boldsymbol{A}_1 z^{n-3} + \boldsymbol{A}_2 z^{n-4} + \cdots + \boldsymbol{A}_{n-2})\boldsymbol{\beta}_2 \\ &\quad + (\boldsymbol{I}z^{n-3} + \boldsymbol{A}_1 z^{n-4} + \boldsymbol{A}_2 z^{n-5} + \cdots + \boldsymbol{A}_{n-3})\boldsymbol{\beta}_3 + \cdots + \boldsymbol{\beta}_n] \end{aligned}$$

$$\begin{aligned}&= (\boldsymbol{I}z^n + \boldsymbol{A}_1 z^{n-1} + \boldsymbol{A}_2 z^{n-2} + \cdots + \boldsymbol{A}_n)^{-1}[(\boldsymbol{I}z^{n-1} + \boldsymbol{A}_1 z^{n-2} + \boldsymbol{A}_2 z^{n-3} + \cdots + \boldsymbol{A}_{n-1})\boldsymbol{\beta}_1\\ &\quad + (\boldsymbol{I}z^{n-2} + \boldsymbol{A}_1 z^{n-3} + \boldsymbol{A}_2 z^{n-4} + \cdots + \boldsymbol{A}_{n-2})\boldsymbol{\beta}_2 + (\boldsymbol{I}z^{n-3} + \boldsymbol{A}_1 z^{n-4} + \boldsymbol{A}_2 z^{n-5} + \cdots + \boldsymbol{A}_{n-3})\boldsymbol{\beta}_3 + \cdots + \boldsymbol{\beta}_n]\\ &= \boldsymbol{A}^{-1}(z)\boldsymbol{B}(z).\end{aligned}$$

Hence, we have the input–output representation:

$$\boldsymbol{A}(z)\boldsymbol{y}(t) = \boldsymbol{B}(z)\boldsymbol{u}(t),$$

where

$$\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \vdots\\ \boldsymbol{\beta}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{I} & & & & \\ \boldsymbol{A}_1 & \boldsymbol{I} & & & \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{I} & & \\ \vdots & & \ddots & \ddots & \\ \boldsymbol{A}_{n-1} & \boldsymbol{A}_{n-2} & \cdots & \boldsymbol{A}_1 & \boldsymbol{I}\end{bmatrix}^{-1} \begin{bmatrix}\boldsymbol{B}_1\\ \boldsymbol{B}_2\\ \vdots\\ \boldsymbol{B}_n\end{bmatrix}.$$

This is equivalent to the difference equation in (1).

6. Examples

Example 1. Consider the following difference equation:

$$\boldsymbol{y}(t) + \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\boldsymbol{y}(t-1) + \begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}\boldsymbol{y}(t-2) = \begin{bmatrix}9 & 10\\ 11 & 12\end{bmatrix}\boldsymbol{u}(t-1) + \begin{bmatrix}13 & 14\\ 15 & 16\end{bmatrix}\boldsymbol{u}(t-2).$$

Using the formula in (9), its observer canonical form is given by

$$\begin{cases}\boldsymbol{x}(t+1) = \begin{bmatrix}-1 & -2 & 1 & 0\\ -3 & -4 & 0 & 1\\ -5 & -6 & 0 & 0\\ -7 & -8 & 0 & 0\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}9 & 10\\ 11 & 12\\ 13 & 14\\ 15 & 16\end{bmatrix}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\boldsymbol{x}(t).\end{cases}$$

Compute the parameters:

$$\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 2 & 1 & 0\\ 3 & 4 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}9 & 10\\ 11 & 12\\ 13 & 14\\ 15 & 16\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & -2 & 1 & 0\\ -3 & -4 & 0 & 1\end{bmatrix}\begin{bmatrix}9 & 10\\ 11 & 12\\ 13 & 14\\ 15 & 16\end{bmatrix} = \begin{bmatrix}9 & 10\\ 11 & 12\\ -18 & -20\\ -56 & -62\end{bmatrix}.$$

Using the formula in (19), its observability canonical form is given by

$$\begin{cases}\boldsymbol{x}(t+1) = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -5 & -6 & -1 & -2\\ -7 & -8 & -3 & -4\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\boldsymbol{x}(t),\end{cases}$$

Example 2. The following observer canonical form:

$$\begin{cases}\boldsymbol{x}(t+1) = \begin{bmatrix}-1 & -2 & 1 & 0\\ -3 & -4 & 0 & 1\\ -5 & -6 & 0 & 0\\ -7 & -8 & 0 & 0\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}9 & 10\\ 11 & 12\\ 13 & 14\\ 15 & 16\end{bmatrix}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\boldsymbol{x}(t),\end{cases}$$

and observability canonical form:

$$\begin{cases} \boldsymbol{x}(t+1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -6 & -1 & -2 \\ -7 & -8 & -3 & -4 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 9 & 10 \\ 11 & 12 \\ -18 & -20 \\ -56 & -62 \end{bmatrix} \boldsymbol{u}(t), \\ \boldsymbol{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) \end{cases}$$

corresponding the difference equation:

$$\boldsymbol{y}(t) + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \boldsymbol{y}(t-1) + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \boldsymbol{y}(t-2) = \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} \boldsymbol{u}(t-1) + \begin{bmatrix} 13 & 14 \\ 15 & 16 \end{bmatrix} \boldsymbol{u}(t-2).$$

7. Conclusions

This paper derives the observer canonical state space model and observability canonical state space model from difference equations and vice versa, by setting different state variables. Two examples are provided to demonstrate the proposed methods.

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